Connections between the map asymptotics constants t_g and some other parameters

Jason Z. Gao
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario K1S5B6
Canada

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Ottawa in Fall

Ottawa in Winter

Maps and rooted maps

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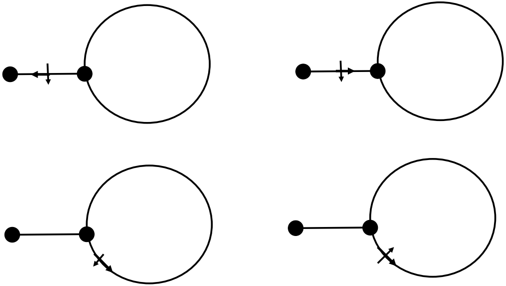
- Two rooted maps are equivalent if there is a homeomorphism from the surface to itself which takes one map to the other and respects the rooting.
- Tutte showed that a rooted map has trivial automorphism group.



The map asymptotics constants

Let $M_g(n)$ be the number of n-edge rooted maps on an orientable surface of genus g. Bender-Canfield (1986) obtained, for each fixed g and as $n \to \infty$,

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More generally we have (Gao, 1993)

$$M_g(n; \mathcal{F}) \sim \alpha t_g (\beta n)^{5(g-1)/2} \gamma^n,$$

where $M_g(n; \mathcal{F})$ is the number of rooted n-edge maps in a given family $\mathcal{F}$ on an orientable surface of genus g, α and β are constants independent of g.

Triangulations and quadrangulations

$$
\begin{aligned}
T_g(n) &\sim 3t_g \left(6^{1/5} n\right)^{5(g-1)/2} (12\sqrt{3})^n, \text{ (triangulations with } 3n \text{ edg}\\
Q_g(n) &\sim 4t_g \left(16^{1/5} n\right)^{5(g-1)/2} 12^n. \text{ (quadrangulations with } 2n \text{ edge}
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$$H_{n,g} = (3n+2)T_g(n+2-2g) \quad \text{for} \quad n \geq 1,$$

$$H_{-1,0} = 1/2, \quad H_{0,0} = 2 \quad \text{and} \quad H_{-1,g} = H_{0,g} = 0 \ \text{ for } \ g \neq 0.$$

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Goulden and Jackson (08) derived the following recursion for $(n,g) \neq (-1,0)$:

$$H_{n,g} = \frac{4(3n+2)}{n+1}\left(n(3n-2)H_{n-2,g-1} + \sum_{i=-1}^{n-1}\sum_{h=0}^{g} H_{i,h}H_{n-2-i,g-h}\right).$$

Connection with Painlevé I

Define

$$u_g = -2^{g-2}\Gamma\left(\frac{5g-1}{2}\right)t_g,\ u(z) = \sum_{g\geq 0} u_g z^{-(5g-1)/2}.$$

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Using the above recursion and the asymptotic formula for $T_g(n)$, Bender-Gao-Richmond (08) derived a quadratic recursion for t_g and a second order ODE for a formal power series defined by t_g. Garoufalidis-Le-Marino (08) observed that our ODE is Painlevé I by a simple transformation, and they showed

$$u_g = \frac{(5g-4)(5g-6)}{48}u_{g-1} - \frac{1}{2}\sum_{h=1}^{g-1} u_h u_{g-h}, \quad u_1 = -\frac{1}{48},$$

and

$$u''(z) = 6u^2(z) - 6z.$$

Asymptotics of t_g

Also

$$u_g \quad \sim \quad -\left(\frac{8\sqrt{3}}{5}\right)^{-2g+1/2} \Gamma(2g-1/2)\frac{3^{1/4}}{2\pi^{3/2}} \times \left(1+\sum_{\ell\geq 1} b_\ell \prod_{k=1}^{\ell} \frac{1}{2g-1/2-k}\right),$$

where b_ℓ can be computed recursively.

Nonorientable map asymptotics constant p_g

There is a nonorientable map asymptotics constant p_g for each nonorientable surface of Euler genus $g = 1 - \chi/2$. Define

$$v_k = 2^{\frac{k-3}{2}}\,\Gamma\left(\frac{5k-1}{4}\right)p_{\frac{k+1}{2}},\ \ v(z) = \sum_{k\geq 0} v_k z^{-(5k-1)/4}.$$

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from which an asymptotic expression of v_g can be obtained. This conjecture has been proved very recently by Carrell (2014).

Connection with rooted binary trees

For a rooted binary tree T, let $L(T)$ and $R(T)$, respectively, denote the left and right subtrees of T. For a rooted binary tree with n vertices, define $X(T)$ recursively by

$$X(T) = X(L(T)) + X(R(T)) + n^2.$$

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$$Y_n := n^{-5/2} X(T),$$

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where T is uniformly distributed among all rooted binary trees with n vertices. Fill and Kapur (04) showed that Y_n has a limit distribution Y whose moments are

$$\mathbf{E}\left(Y^k\right) = \frac{\sqrt{\pi}}{\Gamma((5k-1)/2)} C_k,$$

Connection with rooted binary trees

where C_k satisfies the following recursion

$$C_k = \frac{1}{4}\sum_{j=1}^{k-1}\binom{k}{j}C_jC_{k-j} + \frac{k}{4}(5k-4)(5k-6)C_{k-1},\ C_1 = 1/2.$$

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The following relation is found by Bender-Daalhuis-Gao-Richmond-Wormald (10)

$$t_k = \frac{4}{25k!\Gamma((5k-1)/2)}48^{-k}C_k.$$

Connection with rooted labelled trees

Chapuy (11) gives a bijection between rooted maps and rooted trees such that

$$t_g = \frac{1}{g!\sqrt{\pi}} 2^{(2-5g)/2} \mathbf{E}\left(W^g\right),$$

where the random variable W is defined in terms of the *Integrated Superbrownian Excursion*.

Labelled graphs of a given genus

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$$\frac{G_g(n;k)}{n!} \sim \frac{1}{4}\beta_k^{g-1}\, t_g\, n^{(5g-7)/2} x_k^{-n},$$

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where

$$\begin{array}{lll} x_3 \doteq 0.04751, & x_2 \doteq 0.03819, & x_1 \doteq 0.03673, \\ \beta_3 \doteq 1.48590 \cdot 10^5, & \beta_2 \doteq 7.61501 \cdot 10^4. & \beta_1 \doteq 6.87242 \cdot 10^4. \end{array}$$

Disclaimer

Thank you !

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On page 140 of Bessis-Itzykson-Zuber (1980):

Related problems have recently been studied in the physical literature. We mention here two of them. The first deals with unitary instead of hermitean matrices, and is the work of Gross and Witten [5] and Goldschmidt [6]. The other one considers coupled hermitean matrices. It was studied by two of the present authors [7], and finally solved by Mehta [8].

Our poor knowledge of the mathematical literature does not enable us to quote adequately related work done by mathematicians.